THE CONJUGACY PROBLEM IN WREATH PRODUCTS AND FREE METABELIAN GROUPS

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1. **Introduction.** The conjugacy problem was formulated by Dehn in 1912 for finitely presented groups [1]. The question was whether it can be effectively determined for such a group when two elements are conjugate. This general question was answered in the negative by Novikov in 1954 [6]. But it is still of interest to determine whether the problem is solvable for particular finitely presented groups.

More recently the relation between finitely presented groups and other countable groups has become more fully understood due to the work of Higman [2]. He proved that *a finitely generated group can be embedded in a finitely presented group if (and only if) it has a recursively enumerable set of defining relations.* In such a case, Higman called the group *recursively presented* with respect to the particular set of generators $x_1, \cdots, x_m$ of the group. A finitely generated group G can be written as $gp(x_1, \cdots, x_m; r_1, \cdots)$ where the r_i's lie in the free group F on $x_1, \cdots, x_m$ and where $G \cong F/\bar{R}$ with $\bar{R}$ the normal subgroup of F generated by the r_i's and their conjugates. The elements of such a free group F may be effectively enumerated by considering the lengths of the elements written as words in $x_1, \cdots, x_m$ and arranging elements according to length with the lexicographic ordering by subscipt used for words of the same length. A subset of F such as $\{r_1, r_2, \cdots\}$ is then said to be recursively enumerable if there is some effective enumeration of F with r_i the $f(i)$th element of F for a recursive function f. Such functions can be computed for each integer i and for any integer n it can be determined whether there is an i such that $f(i) = n$. (For a complete discussion of recursive functions, the reader is referred to [3].) Because of Higman's work, we see that recursively presented groups are actually as well presented as the finitely presented groups. We may then speak of the conjugacy problem for particular groups of this type also. Let us say that a group G is of "class $\mathscr{C}$" if it is recursively presented and has a solvable conjugacy problem for the given presentation. Thus the class $\mathscr{C}$ actually represents a class of presentations of groups rather than of groups per se.

The free groups of finite rank and the finitely generated free Abelian groups are of class $\mathscr{C}$. This suggests that it might be worthwhile to investigate the con-

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jugacy problem for certain *relatively free groups*, that is for groups free in some variety. In particular, we might look at the product variety of two Abelian varieties. This brings us to the question of whether the finitely generated free metabelian groups, groups which are recursively presented, are of class $\mathscr{C}$. The main result of this paper is the proof of the following:

THEOREM A. *Finitely generated free metabelian groups are of class* $\mathscr{C}$.

In view of the close connection between product varieties and wreath products and also because of the seeming intractability of approaching the conjugacy problem directly in such a free metabelian group M, we shall make use of the known effective embedding of M in a wreath product of two free Abelian groups [7], [4]. Our procedure for proving Theorem A will be the following:

First, in §2, we give a construction of the wreath product of two recursively presented groups, noting that this wreath product is itself recursively presented in the case that the original groups are both of class $\mathscr{C}$.

Next, using arguments similar to those of Peter Neumann [5], we prove in §3 the following:

THEOREM B. *The* (*restricted*) *wreath product* $W = A \text{ wr } B$ *of two nontrivial groups of class* $\mathscr{C}$ *is itself of class* $\mathscr{C}$ *if and only if the group* B *has a solvable power problem.*

Here we understand by the solution of the power problem in the recursively presented group B the ability to decide for any two elements x and y in B whether there exists an integer n such that $y = x^n$.

Finally, in §4, we make use of a mapping ε of the finitely generated free metabelian group M onto a subgroup $M\varepsilon = \boldsymbol{M}$ of a wreath product W of class $\mathscr{C}$ and prove the following:

THEOREM C. *Two elements* x *and* y *in* M *are conjugate in* M *if and only if their images* $x\varepsilon$ *and* $y\varepsilon$ *are conjugate in* W.

2. **Preliminaries**. Given two elements g and h in a group G, we denote by $[g,h]$ the commutator $g^{-1}h^{-1}gh$. Let A be a group and S a nonempty set; we define the Cartesian power A^S as $\{f \mid f: S \to A\}$. If f and g lie in A^S, then so does fg and for all s in S we have $fg(s) = f(s)g(s)$. If B is a group of permutations on S, B is isomorphic to a group of automorphisms of A^S. For b in B and f in A^S, let f^b denote the image of f under the automorphism corresponding to b, then $f^b(s) = f(s^{b^{-1}})$ for all s in S. We write $s^{b^{-1}}$ for the image under b^{-1} of s. Let $\bar{W}$ be the splitting extension of A^S by the group B so that $\bar{W} = \{(b,f) \mid b \in B, f \in A^S\}$ were we define multiplication in $\bar{W}$ by $(b,f)(c,g) = (bc, f^c g)$ for b and c in B and f and g in A^S. Let 1 denote the element of A^S, such that $1(s) = 1$ in A for all s in S. Then $\{(b,1)\}$ for b in B is a subgroup of $\bar{W}$ isomorphic to B and we shall identify it with B. Similarly $\{(1,f)\}$ with f in A^S shall be identified with A^S. Then A^S is normal in $\bar{W}$ and complemented by B.

Elements of $\bar{W}$ thus factorize uniquely as products of the form bf with b in B and f in A^S. The automorphisms of A^S induced by b in B are the restrictions to A^S of the inner automorphisms of $\bar{W}$ induced by transformation by b. We obtain the unrestricted or standard wreath product $A\mathrm{Wr}B$ of A by B when we take the set S as B itself and the action of B given by multiplication on the right so that for f in A^B and b in B, f^b is given in terms of f by the relation $f^b(\beta) = f(\beta b^{-1})$. We obtain the *restricted* wreath product $W = A\,\mathrm{wr}\,B$ with which we shall be concerned in this paper by considering the direct product $A^{(B)}$ rather than the Cartesian product A^B. We denote by $\sigma(f)$ the support of f, that is $\{b \in B \mid f(b) \neq 1\}$. We may label the coordinate subgroups of K by the elements of B so that if $b \in B$, A_b is $\{f \in K \mid f(\beta) = 1 \text{ if } \beta \neq b\}$; that is $A_b = \{f \in K \mid \sigma(f) \subset \{b\}\}$. There is a natural isomorphism υ_b: $A_b \to A$ given by $\upsilon_b(f) = f(b)$ for all f in A_b.

Consider now two *recursively presented* groups A and B, then:

$$W = A\,\mathrm{wr}B = gp(a_1, a_2, \cdots, a_m,\ b_1, b_2, \cdots, b_n \mid r(a_1, \cdots, a_i) = 1 \text{ for } r \text{ in } R,$$

$$t(b_1, \cdots, b_j) = 1 \text{ for } t \text{ in } T,$$

and $[w(b_1, \cdots, b_k)^{-1} a_p w(b_1, \cdots, b_k), a_q] = 1$

for all a_p and a_q and all w such that $w(b_1, \cdots b_k) \neq 1)$

where R and T are the sets of relators of the groups A and B resp. Thus W itself is recursively presented in the case that A is Abelian or that B has a solvable word problem (in which case the words $w(b_1, \cdots, b_k)$ which are nontrivial elements of B may be effectively enumerated). This is certainly true if A and B are of class $\mathscr{C}$. Clearly the solvability of the conjugacy problem in B is necessary for the solvability of the conjugacy problem in $W = A\mathrm{wr}B$ (or in any splitting extension of an arbitrary group K by the group B) since the existence of an element z in W of the form dh such that $zx = yz$ with $x = bf$ and $y = cg$ implies that $db = cd$ in B. The necessity of the solvability of the conjugacy problem in A for its solvability in W follows from the fact that two elements f and g in $K = A^{(B)}$ such that $\sigma(f)$ and $\sigma(g)$ are both equal to $\{b\}$ for some b in B are conjugate in W if and only if $f(b)$ and $g(b)$ are conjugate in A.

3. The conjugacy problem in restricted wreath products.

3.1. *Outline of the proof of Theorem* B. We recall that Theorem B states that the wreath product $W = A\mathrm{wr}B$ of two nontrivial groups of class $\mathscr{C}$ is itself of class $\mathscr{C}$ if and only if the group B has a solvable power problem. The proof of this theorem consist of a reduction of the conjugacy problem in W to conjugacy poblems in A and B. In order to accomplish this reduction, we shall use mappings of $K = A^{(B)}$ into A. Given any element b in B, let $T = \{t_i\}$ be a particular set of coset representatives for $gp(b)$ in B. We associate with T classes of maps $\{\pi_i^{(\gamma)}\}$ for each γ in B as follows:

$$\pi_i^{(\gamma)}(f) = \begin{cases} \prod_{j=0}^{N-1} f(t_i b^j \gamma^{-1}) & \text{for } b \text{ of finite order } N \\ \prod_{j=-\infty}^{\infty} f(t_i b^j \gamma^{-1}) & \text{for } b \text{ of infinite order.} \end{cases}$$

We note that these products are always finite ones since K is a direct rather than a Cartesian product. For simplicity we shall write π_i for $\pi_i^{(1)}$ where 1 is the identity in B.

Two elements $x = bf$ and $y = cg$ in W are conjugate if and only if there exists an element z of the form dh in W such that $zx = yz$. Here b, c, and d lie in B and f, g, and h lie in K. The relation $zx = yz$ in W is equivalent to the pair of relations $db = cd$ in B and $g^d = h^b f h^{-1}$ in K.

3.2. Lemma. *Let $W = A\mathrm{wr}B$ and let b and d in B and f, g, and h in $K = A^{(B)}$ be such that $g^d = h^b f h^{-1}$. Then for all cosest representatives t_i in T and all integers m and n with $n \geqq 0$, we have*:

$$\prod_{j=m}^{m+n} g(t_i b^j d^{-1}) = h(t_i b^{m-1}) \left[\prod_{j=m}^{m+n} f(t_i b^j) \right] h^{-1}(t_i b^{m+n}). \tag{3.2.1}$$

Proof. By hypothesis, we have for any integer j that:

$$g^d(t_i b^j) = g(t_i b^j d^{-1}) = h(t_i b^{j-1}) f(t_i b^j) h^{-1}(t_i b^j).$$

The lemma then follows upon taking the product over j.

3.3 Corollary. *Let $x = bf$, $y = cg$, and $z = dh$ in $W = A\mathrm{wr}B$ be such that $zx = yz$. Then, if b is of finite order N, $\pi_i^{(d)}(g)$ is conjugate in A to $\pi_i(f)$ for all i.*

Proof. We need only set $m = 0$ and $n = N - 1$ in line (3.2.1) and apply the definitions of the functions π_i and $\pi_i^{(d)}$.

3.4. Corollary. *Let $x = bf$, $y = cg$ and $z = dh$ in $W = A\mathrm{wr}B$ be such that $zx = yz$. Then, if b is of infinite order, $\pi_i^{(d)}(g)$ is equal in A to $\pi_i(f)$ for all i.*

Proof. Since K is a direct product, there exists for any i a pair of integers M and N with $N \geqq 0$ such that:

$$\pi_i^{(d)}(g) = \prod_{j=M}^{M+N} g(t_i b^j d^{-1}) \text{ and } \pi_i(f) = \prod_{j=M}^{M+N} f(t_i b^j).$$

Thus it follows from (3.2.1) that $\pi_i^{(d)}(g) = h(t_i b^{M-1})\, \pi_i(f) h^{-1}(t_i b^{M+N})$. But $h(t_i b^{M-1})$ and $h(t_i b^{M+N})$ must be the identity in A for otherwise applications (3.2.1) with $m = M - k$, $n = N + k$ and with $m = M$, $n = N + k$ for all integers $k \geqq 0$ would yield the contradiction that $h(\beta)$ is different from 1 for infinitely many β in B.

3.5. Proposition. *Two elements $x = bf$ and $y = cg$ of $W = A\mathrm{wr}B$ are conju-*

gate in W where b is of infinite order if and only if there exists an element d in B such that for all i:

(3.5.1)
(1) $db = cd$ *in* B,
(2) $\pi_i^{(d)}(g)$ *is equal to* $\pi_i(f)$ *in* B.

Proof. Since x and y are conjugate in W if and only if there is a z in W of the form dh such that $zx = yz$, the necessity of (3.5.1) follows from Corollary 3.4. Assume then that (3.5.1) holds for some d. We construct h so that for $z = dh$ we have $zx = yz$ as follows: Set

$$h(t_ib^k) = \left[\prod_{j \leqq k} g(t_ib^jd^{-1})\right]^{-1}\left[\prod_{j \leqq k} f(t_ib^j)\right]$$

(for all integers k and t_i in T).

3.6. Proposition. *Two elements $x = bf$ and $y = cg$ of $W = A\mathrm{wr}B$ are conjugate in W where b is of finite order N if and only if there exists an element d in B such that for all i:*

(3.6.1)
(1) $db = cd$ *in* B,
(2) $\pi_i^{(d)}(g)$ *is conjugate to* $\pi_i(f)$ *in* A.

Proof. The necessity of (3.6.1) follows easily from Corollary 3.3. Assume then that (3.6.1) holds and let α_i be such that $\pi_i^{(d)}(g) = \alpha_i\pi_i(f)\alpha_i^{-1}$. We construct h so that for $z = dh$ we have $zx = yz$ as follows: Set

$$h(t_ib^k) = \left[\prod_{j=0}^{k} g(t_ib^jd^{-1})\right]^{-1}\alpha_i\left[\prod_{j=0}^{k} f(t_ib^j)\right]$$

(for all t_i in T and $k = 0,1,\cdots N-1$).

It is now a simple matter to prove Theorem B.

3.7. Theorem B. *The wreath product $W = A\mathrm{wr}B$ of two nontrivial groups of class $\mathscr{C}$ is itself of class $\mathscr{C}$ if and only if the group B has a solvable power problem.*

Proof of necessity. Let b and c be elements of B and let a in A be different from the identity. Define the element k in $K = A^B$ as follows:

$$k(\beta) = \begin{cases} a & \text{for } \beta = 1, \\ a^{-1} & \text{for } \beta = c, \\ 1 & \text{otherwise.} \end{cases}$$

Then b is conjugate in W to bk if c is a power of b in B.

Proof of sufficiency. Assume now that A and B are both of class $\mathscr{C}$ and that

B has a solvable power problem. Given any two elements $x = bf$ and $y = cg$ in W, we must show that it is possible to decide *in a finite number of steps* whether there is a d in B such that the appropriate pair of relations (3.5.1) or (3.6.1) holds. We exhibit an algorithm for this as follows:

Step I. We use the fact that B is of class $\mathscr{C}$ to decide whether there is a d in B such that $db = cd$. If no such d exists, then x and y are not conjugate in W. If such d exists, we proceed to

Step II. Since B has a solvable power problem we can determine the order of b in B. We may also partition the support $\sigma(f)$ into equivalence classes in such a way that two elements of $\sigma(f)$ lie in the same equivalence class if and only if they lie in the same left coset of $gp(b)$ in B. For *these* cosets we chose a corresponding *finite* set of representatives $\{t_i\}$.

Case (1). $g = 1$. Then x is conjugate to $y = c$ if and only if all $\pi_i(f) = 1$. This is clearly decidable.

Case (2). $g \not\equiv 1$ but all $\pi_i(f) = 1$. In this case x and y are conjugate if and only if all $\pi_i^{(d)}(g) = 1$. Since, by Step I, we have assumed that there exists a d in B such that $db = cd$, we have for all i: $g(t_i b^j d^{-1}) = g(t_i d^{-1} c^j)$ with j an arbitrary integer. We now show that it can be decided whether all $\pi_i^{(d)}(g) = 1$ *using only the existence* of a d in B such that $db = cd$. Just as we partitioned $\sigma(f)$ using left cosets of $gp(b)$ in B, we may we may partition $\sigma(g)$ using left cosets of $gp(c)$ in B. Then we construct maps $\bar{\pi}_i$ with $\bar{\pi}_i(g) = \pi_j g(s_i c^j)$ for s_i in a set S of coset representatives of $gp(c)$ in B. Here again j runs from 0 through $N-1$ if b (and thus also c) is of order N and from $-\infty$ to ∞ otherwise. One such set S consists of $\{t_i d^{-1}\}$ for a fixed d. But the product in $\bar{\pi}_i$ is 1 for one set S if and only if it is 1 for all sets, since changing S only results in a cyclic permutation of the factors in the product. Thus x and y are conjugate if and only if all $\bar{\pi}_i(g) = 1$ and there exists an element d in B such that $db = cd$.

Case (3). $g \not\equiv 1$ and some $\pi_i(f) \neq 1$. Let k be a fixed integer such that $\pi_k(f) \neq 1$. In order for there to exist a d in B such that each $\pi_i(f)$ is equal to (resp. conjugate to) the corresponding $\pi_i^{(d)}(g)$, there must in particular be a d in B such that $\pi_k^{(d)}(g)$ is different from 1. Thus also there is an integer n such that $g(t_k b^n d^{-1})$ is different from 1. Let $\sigma(g) = \{\beta_1, \beta_2, \cdots, \beta_m\}$. To determine whether x and y are conjugate in W, we need only test for $d = \beta_1^{-1} t_k b^n, \cdots, \beta_m^{-1} t_k b^n$ whether $db = cd$ and $\pi_i^{(d)}(g)$ equals (resp. is conjugate to) $\pi_i(f)$ for all i. This testing process may possibly be shortened by observing that if $d_1 = d_2 b^r$ for some integer r then $d_1 b = cd_1$ if and only if $d_2 b = cd_2$ and $\pi_i^{(d_1)}(g)$ equals (is conjugate to) $\pi_i(f)$ if and only if $\pi_i^{(d_2)}(g)$ equals (is conjugate to) $\pi_i(f)$.

We note that, in the case that A and B are both free Abelian groups of finite rank, $W = A \operatorname{wr} B$ is of class $\mathscr{C}$.

4. **The conjugacy problem in finitely generated free metabelian groups.** Let M be the free metabelian group freely generated by $\{x_1, \cdots, x_n\}$ with n finite. Thus M

is an isomorphic copy of an absolutely free group F of rank n by its second derived group F'': $M \cong F/F''$. It follows from theorems of Smel'kin and Magnus that, if

$$A = gp(a_1, \cdots, a_n \mid [a_i, a_j] = 1 \text{ for all } i \text{ and } j = 1, \cdots, n)$$

and

$$B = gp(b_1, \cdots, b_n \mid [b_i, b_j] = 1 \text{ for all } i \text{ and } j = 1, \cdots, n)$$

are free Abelian groups of rank n, then $W = A\mathrm{wr}B$ contains a copy $\mathbf{M}$ of M. Indeed $\mathbf{M}$ is generated by the elements $m_i = b_i f_i$ in W where the f_i in $K = A^{(B)}$ are defined by:

$$f_i(\beta) = \begin{cases} a_i & \text{for } \beta = b_i \text{ with } i = 1, \cdots, n, \\ 1 & \text{otherwise} \end{cases}$$

4.1. *Outline of the proof of Theorem* C. We recall that Theorem C states that two elements x and y in M are conjugate in M if and only if their images $x\varepsilon$ and $y\varepsilon$ are conjugate in W. We shall now identify M with the embedded group and write x and y for the images in W. The proof of the theorem naturally breaks up into two parts. First we prove by Proposition 4.2 that two elements of M', the derived group of M, are conjugate in M if and only if they are conjugate in W. For the second part of the proof we deal with elements outside M' and show that if there is an element z in W such that $zx = yz$ with x and y in M *but not in* M' then z must lie in M. In order to accomplish this, we need to look more closely at the position of M' in W.

W is a splitting extension of $K = A^{(B)}$ by an Abelian group B and thus $W' \subset K$ and so also $M' \subset K$. Because A too is Abelian, $(\beta f)^{-1} k(\beta f) = \beta^{-1} k \beta$ for all β in B and all f and k in K. Since in addition $MK = W$, any element w in W may be written in the form $m_1^{v_1} \cdots m_n^{v_n} k$ with k in K for some set of integers $v_1, \cdots, v_n$; thus w lies in M if and only if k lies in M'. By means of Lemmas 4.6 through 4.8, we shall be able to show that if kk^{-b} lies in M' where k is in K and where $b \neq 1$ in B then k is itself in M'. Once we have this, Theorem C follows quite easily.

4.2. PROPOSITION. *Given any group W with an Abelian normal subgroup K and a subgroup M such that M generates W modulo K, that is such that $MK = W$, then two elements x and y in $M \cap K$ are conjugate in M if and only if they are conjugate in W.*

Proof. Since M is a subgroup of W, we need only show that if x and y in $M \cap K$ are conjugate in W then they are conjugate in M itself. By the assumption of conjugate in W, there is an element w in W of the form mk with m in M and k in K such that $y = (mk)^{-1} x(mk)$. Thus $k^{-1}(m^{-1}xm)k = y$. But K is normal in W and so $m^{-1}xm$ lies in K. Furthermore K is Abelian and thus $m^{-1}xm = y$ or x and y are conjugate in M.

An immediate consequence of this proposition is the following:

4.3. Corollary. *If $W = A\mathrm{wr}B$ where A and B are free Abelian groups of rank n, then two elements of M' are conjugate in M, the embedded free metabelian group, if and only if they are conjugate in W.*

The transform in M of the commutators $[m_i, m_j]$ with $i > j$ generate M'. Let $[m_i, m_j]$ be denoted by $d_{i,j}$. It then follows that:

$$d_{i,j}(\beta) = \begin{cases} a_j & \text{for } \beta = b_j, \\ a_i^{-1} & \text{for } \beta = b_i, \\ a_i a_j^{-1} & \text{for } \beta = b_i b_j, \\ 1 & \text{otherwise.} \end{cases}$$

In order to test whether an element k in K lies in M', we shall need mappings of K into A similar to those used for more general wreath products in §3. Any element in the free Abelian group B may be written in a unique normal form as $b_1^{\mu_1} \cdots b_n^{\mu_n}$ for some set of integers $\mu_1, \cdots, \mu_n$. We introduce in B equivalence relations e_l for each integer l with $1 \leqq l \leqq n$ as follows: $\beta_1 = b_1^{\mu_{11}} \cdots b_n^{\mu_{n1}}$ lies in the same e_l-equivalence class as $\beta_2 = b_1^{\mu_{12}} \cdots b_n^{\mu_{n2}}$ if μ_{i_1} is equal to μ_{i2} for all $i \leqq l$. We partition B into such equivalence classes $S_\upsilon^{(l)}$ for any fixed integer l with $1 \leqq l \leqq n$. The mappings we shall need are the mappings $\pi_\upsilon^{(l)}$ defined by $\pi_\upsilon^{(l)}(f) = \prod_{\beta \in S_\upsilon(l)} f(\beta)$. These maps are then uniquely determined since the group A is Abelian.

4.4. Definition. An element α of the free Abelian group A generated by $\{a_1, \cdots, a_n)$ will be said to be "free of $a_1, \cdots, a_l$" if, in its canonical representation $\alpha = a_1^{\gamma_1} \cdots a_n^{\gamma_n}$, $\gamma_1 = \gamma_2 = \cdots = \gamma_l = 0$.

4.5. Definition. An element k in $K = A^{(B)}$ will be said to be "free of $a_1, \cdots, a_l$" if $k(\beta)$ is free of $a_1, \cdots, a_l$ for all β in B.

4.6. Lemma. *Let m in M' be a word in the transforms of only those $d_{i,j}$ with $i > j \geqq l$ for some fixed integer $l \geqq 1$. Then $\pi_\upsilon^{(l)}(m)$ is free of $a_1, \cdots, a_l$ for all υ.*

Proof. From the definitions of the elements $d_{i,j}$, it is obvious that m is free of $a_1, \cdots, a_{l-1}$. Only $d_{i,l}$ can involve a_l and for any υ these occurrences of a_l cancel in $\pi_\upsilon^{(l)}(m)$.

4.7. Lemma. *Let k in K be free of $a_1, \cdots, a_{l-1}$ for some fixed $l \geqq 1$. Then if $\pi_\upsilon^{(l)}(k)$ is free of $a_1, \cdots, a_{l-1}$ and a_l for all υ, there exists an element m in M' (free of $a_1, \cdots, a_{l-1}$) such that km^{-1} is free of $a_1, \cdots, a_l$.*

Proof. We begin with a series of reductions which enable us to look at only those k which are of a very simple form. Since transformation by the element $m_1^{\gamma_1} \cdots m_n^{\gamma_n}$ is equivalent to transformation by the element $b_1^{\gamma_1} \cdots b_n^{\gamma_n}$, we may look at a transform of k rather than k itself. It is enough to prove the lemma when $\sigma(k) \subset S_v^{(l)}$ for some υ. We thus assume that $\sigma(k)$ is actually $\{\beta_1, \beta_2, \cdots, \beta_r\}$ where each $\beta_\rho = b_l b_{l+1}^{e_{l+1}(\rho)} \cdots b_n^{e_n(\rho)}$ for $\rho = 1, \cdots, r$ and where $\beta_\rho < \beta_{\rho+1}$. Next we write

k as a product gh where for $k(\beta) = a_l^{\mu_l} a_{l+1}^{\mu_{l+1}} \cdots a_n^{\mu_n}$ we set $g(\beta) = a_l^{\mu_l}$ and $h(\beta) = a_{l+1}^{\mu_{l+1}} \cdots a_n^{\mu_n}$. We need only prove the lemma for g.

The element g in K may itself be written as the product $g_1^{\tau(1)} \cdots g_{r-1}^{\tau(r-1)}$ where if $g(\beta_\rho) = a_l^{\mu(\rho)}$ we set:

$$g_\rho(\beta) = \begin{cases} a_l & \text{for } \beta = \beta_\rho, \\ a_l^{-1} & \text{for } \beta = \beta_{\rho+1}, \\ 1 & \text{otherwise}, \end{cases}$$

with $\tau(\rho) = \mu(1) + \mu(2) + \cdots + \mu(\rho)$.

We need only consider one such element say f in K defined by:

$$f(\beta) = \begin{cases} a_l & \text{for } \beta = b_l, \\ a_l^{-1} & \text{for } \beta = b_l b_{l+1}^{\sigma_{l+1}} \cdots b_n^{\sigma_n}, \\ 1 & \text{otherwise}. \end{cases}$$

A further simplification is accomplished by writing f as a product $f_l \cdots f_{n-1}$ with:

$$f_{l+j}(\beta) = \begin{cases} a_l & \text{for } \beta = b_l b_{l+1}^{\sigma_{l+1}} \cdots b_{l+j}^{\sigma_{l+j}}. \\ a_l^{-1} & \text{for } \beta = b_l b_{l+1}^{\sigma_{l+1}} \cdots b_{l+j}^{\sigma_{l+j}} b_{l+j+1}^{\sigma_{l+j+1}}, \\ 1 & \text{otherwise}, \end{cases}$$

for $j = 0, 1, \cdots, n - l - 1$.

By setting $\bar{f}$ equal to some particular f_{l+j}, $s = l + j + 1$, $r = \sigma_{l+j+1}$, and $\beta_0 = b_{l+1}^{\sigma_{l+1}} b_{l+2}^{\sigma_{l+2}} \cdots b_{l+j}^{\sigma_{l+j}}$, we have:

$$\bar{f}(\beta) = \begin{cases} a_l & \text{for } \beta = b_l \beta_0, \\ a_l^{-1} & \text{for } \beta = b_l \beta_0 b_s^r \quad \text{with } r > 0, \\ 1 & \text{otherwise}. \end{cases}$$

Finally setting $\bar{f}$ equal to the product $\bar{f}_0 \bar{f}_1 \cdots \bar{f}_{r-1}$ where

$$\bar{f}_i(\beta) = \begin{cases} a_l & \text{for } \beta = b_l \beta_0 b_s^i, \\ a_l^{-1} & \text{for } \beta = b_l \beta_0 b_s^{i+1}, \\ 1 & \text{otherwise}, \end{cases}$$

we prove the lemma by associating with each $\bar{f}_i$ the transform of $d_{s,l}$ by $(\beta_0 b_s^i)^{-1}$.

4.8. Lemma. *Let k in K and $b = b_1^{\gamma_1} \cdots b_n^{\gamma_n}$ in B with not all of $\gamma_1, \cdots, \gamma_l = 0$ be such that for $t = kk^{-b}$ the product $\pi_\upsilon^{(l)}(t)$ is free of $a_1, \cdots, a_l$ for all υ. Then $\pi_\upsilon^{(l)}(k)$ is free of $a_1, \cdots, a_l$ for all υ.*

Proof. Assume the lemma to be false. Then there exists a particular υ such that

$\pi_\upsilon^{(l)}(k) = a_1^{\mu_1} \cdots a_n^{\mu_n}$ with not all of $\mu_1, \mu_2, \cdots, \mu_l = 0$. Associated with this product, there is then an equivalence class $S_\upsilon^{(l)}$ that intersects $\sigma(k)$ nontrivially. Let us denote this class by S. Then, since $\pi_\upsilon^{(l)}(t)$ is free of $a_1, \cdots, a_l$, the set S_b defined as $\{\beta' \in B \mid \beta' = \beta b \text{ for } \beta \text{ in } S\}$ also intersects $\sigma(k)$ nontrivially. For if the product of k taken over the set S_b is $a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n}$ we would have $\varepsilon_i = -\mu_i$ for all $i \leqq l$. Thus, for infinitely many j, the sets S_{b_j} must intersect $\sigma(k)$ nontrivially. This contradicts the fact that k lies in a *direct* product K. The lemma thus follows.

4.9. COROLLARY. *If m is a word in M' in the transforms of only those $d_{i,j}$ with $i > j \geqq l$ and if $kk^{-b} = m$ where b in B equals the product $b_1^{\gamma_1} \cdots b_n^{\gamma_n}$ with not all of $\gamma_1, \cdots, \gamma_l = 0$, then $\pi_\upsilon^{(l)}(k)$ is free of $a_1, \cdots, a_l$.*

Proof. By Lemma 4.6 we know that $\pi_\upsilon^{(l)}(m)$ is free of $a_1, \cdots, a_l$ for all υ. Thus we need only apply Lemma 4.7.

4.10. THEOREM C. *Two elements x and y in M are conjugate in M if and only if their images $x\varepsilon$ and $y\varepsilon$ are conjugate in W.*

Proof. We may assume that M itself is embedded in W. Let x be of the form bf in W, y of the form cg in W, then we need only prove that if x and y are conjugate in W they are conjugate within the subgroup M itself. Using the solution of the conjugacy problem in wreath products and the algorithm presented in §3, we have for x and y conjugate in W that b is conjugate to c in B. But B is Abelian and thus b is equal to c.

Case (1). $b = 1$. In this case $x = f$ and $y = g$ lie in $M \cap K$. But $M \cap K$ is just M' and so by Corollary 4.3 if x and y are conjugate in W, then they are conjugate in M itself.

Case (2). $b \neq 1$. In this case $x = bf$ and $y = bg$ where $b = b_1^{\gamma_1} \cdots b_n^{\gamma_n}$ with some γ_i different from 0. By renumbering the elements b_i of B, if necessary, we may assume that $\gamma_1 \neq 0$. By hypothesis there is a z in W of the form dh such that $zx = yz$. We now write the elements x, y, and z as $m_1^{\gamma_1} \cdots m_n^{\gamma_n} r$, $m_1^{\gamma_1} \cdots m_n^{\gamma_n} s$ and $m_1^{\sigma_1} \cdots m_n^{\sigma_n} k$ respectively in $W = MK$. Here r and s lie in M' and k lies in K. *Our aim then is to prove that k lies in M'.* For simplicity of notation we write $x = qr$, $y = qs$, and $z = pk$.

Writing out the identity $zx = yz$, we have $pkqr = qspk$ or that $pqk^q r = qps^p k$. But then $pqk^b r = qps^d k$ and we obtain the relation:

$$kk^{-b} = [p, q] rs^{-d}. \tag{4.10.1}$$

But $[p, q] rs^{-d}$ lies in M' and thus kk^{-b} must also lie in M'. Let now $t = kk^{-b}$. Then, by Lemma 4.6 applied for $l = 1$, we have $\pi_\upsilon^{(1)}(t)$ free of a_1. Thus by Lemma 4.8 $\pi_\upsilon^{(1)}(k)$ is also free of a_1. Therefore, by Lemma 4.7, there exists an element m in M' such that km^{-1} is free of a_1. But since m is in M', so $m^{-b} = m^{-q}$. Dividing through by mm^{-b} in the relation (4.10.1), we obtain the relation:

(4.10.1)' $$(km^{-1})(km^{-1})^{-b} = [p,q]rs^{-d}(mm^{-b})^{-1}.$$

Setting $k_1 = km^{-1}$ we proceed in exactly the same manner to obtain an element m_1 in M' such that $k_1m_1^{-1}$ is free of both a_1 and a_2. A simple induction argument provides us at the $(n-1)$st stage with an expression for k as a product of elements in M'. Thus k is in M' and the theorem is proved in this case also.

4.11. THEOREM A. *Finitely generated free metabelian groups are of class* $\mathscr{C}$.

Proof. We have only to combine Theorems B and C.

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